# THE TEACHING OF MATHEMATICS IN THE ELEMENTARY AND THE SECONDARY SCHOOL.*

BY ARTHUR SULLIVAN GALE.

During recent years there has developed in this country a very real interest in the *teaching* of mathematics, as evidenced by the formation of numerous associations of teachers of mathematics. This interest may be traced to two principal sources. The great mathematical revival finding its expression in the rapid and healthy growth of the American Mathematical Society has had naturally a reactionary effect upon collegiate and then upon secondary instruction. This effect is seen in the effort to put upon a scientific basis the elementary parts of mathematics in their relation to the subject as a whole. At the same time, modern pedagogy holds as its principal thesis that both subject matter and manner of presentation must be arranged with reference to a psychological study of the pupil. Hence a further rearrangement of mathematical material is required with the express object of obtaining and retaining the student's interest. These two courses of the demand for improvement in mathematical instruction call for two lines of preparation on the part of the teacher, mathematical and pedagogical. A man may have studied a wide range of mathematical topics and yet have so poor a notion of how to present his ideas that it takes several years' experience to learn to teach; and in the meantime many of his pupils may discover, or believe that they have discovered, that they are so mentally deficient as to be unable to grasp mathematics. On the other hand, a man may learn something of the technique of teaching and be so ignorant of the principles of the science that his students do not obtain any idea of the spirit of mathematical studies. Such ignorance may be partially pardoned in the man who is forced to teach many different subjects; but it is, even at present, no novelty to find a teacher of mathematics only, who thinks that he requires his students to give a complete reason for every step in a geometrical demonstration. It is generally conceded that the normal schools have been unable to afford proper mathematical training for the high school teacher, and it is gratifying that the colleges are beginning to

* By J. W. A. Young. Longmans, Green and Co., 1907.

offer courses in mathematics arranged especially for those intending to enter the field of secondary instruction. Such courses should touch on many topics having an immediate bearing on elementary algebra and geometry and not ordinarily included in the courses usually offered to undergraduates, as well as some discussion of pedagogical principles. The proper person to conduct them has been aptly described by a well-known mathematician as "a mathematician sufficiently interested in his subject to publish occasional investigations, and who has the *pedagogic instinct.*"

There is no lack of literature on the pedagogy of mathematics in various periodicals, but the number of texts is so limited that Professor Young's book should be welcomed by all teachers of secondary mathematics. As indicated in the preface, the historical aspect of the subject is not considered, nor are topics of subject matter taken up except in illustrating various forms of presentation. Yet the last five chapters (pp. 189–346) contain many valuable suggestions relating to subject matter.

Chapter I, the study of the pedagogy of mathematics, is chiefly introductory, the first important discussion being in Chapter II, the purpose and value of the study of mathematics (pp. 9–52). It is the best and most comprehensive treatment of the topic which has come to the attention of the reviewer. While the author gives due prominence to the practical value of mathematics, considering the importance of its facts and its relations to nature and to general culture, he places chief emphasis on mathematics as a mode of thought of the utmost importance. "Throughout all mathematics, from the first numbers lisped in the nursery to the aged mathematician's last sigh, the chief end of mathematics is thought, not routine,—*natural* thought, exercising the powers of the thinker in an unforced and interested manner . . ." (p. 204). Pierce's definition, "Mathematics is the science of necessary conclusions," is used and a number of examples of reasoning of daily occurrence are examined in detail. There follows a brief consideration of eleven other functions of mathematics, to which might be added its disciplinary value in requiring continuous application throughout long intervals of time. It is the occasional relaxing and consequent falling behind which causes many of the failures in mathematics.

In an analysis of the book it is natural to associate the next

five chapters, whose headings are: Methods and modes, The heuristic method, The individual mode, The Perry movement: the laboratory method, and Miscellaneous points of method and mode. In the first of these, method is defined as the manner in which the subject matter is arranged and developed, and mode as the manner in which it is presented to the pupils. The methods considered are the synthetic, analytic, deductive, inductive, Socratic, heuristic, and laboratory, with emphasis on the pedagogic importance of analysis and induction. The examination, recitation, lecture, genetic, heuristic, individual and laboratory modes are explained, and in the discussion of their relative values we read that "The test of any, of every mode, is whether or not it is, at the time when it is employed, the mode which enables the teacher to give to the class, to every pupil, the most of himself, of his knowledge, of his experience, of his guidance, of his enthusiasm, of his inspiration" (p. 67). This sentence shows, perhaps as well as any other, the broad-minded point of view of the author. Throughout these chapters he presents the disadvantages as well as the advantages of a particular mode or method, so that the reader, to derive benefit from his perusal of the book, is forced to think for himself, a process as important for the teacher as for the pupil. The chapter on the laboratory method contains many helpful suggestions which can be used, and are being widely used, even by those who do not regard their class rooms as laboratories.

One point, perhaps, deserves especial mention, namely, the advocacy of the "free use of motion" in geometrical proofs. The reviewer would go even further and plead for a more consistent use of geometric transformations in both elementary and higher geometry. With very little of the technique of transformations a great deal may be accomplished, and much use may profitably be made not only of displacements but also of simple symmetry and similitude transformations. Recent foreign texts have made much of symmetry, and the homothetic transformation, illustrated by the pantograph (there should be one in every class-room in geometry), might well be made the basis of the consideration of similar figures. The theory of transformations underlies the whole domain of geometry in so fundamental a manner that a student may be introduced to many of the most far-reaching theorems of geometry through a study of the es-

sential properties of a few comparatively simple transformations. But while a free use of motion is desirable, the example used in slightly different form in two places (pp. 110 and 275) is unfortunate. It is the Playfair-Thibaut* proof (?) that the sum of the angles of a triangle is two right angles. Are we displaying "absolute truth toward the pupil" if we ask him to accept a proof which was believed by its originators to have answered the question associated for centuries with Euclid's parallel postulate?

If the answer to this question depended only upon the relation of non-Euclidean geometry to elementary instruction it would not be worth discussing, but it involves no less than a consideration of the general principles upon which the course in geometry is to be arranged. Professor Young's ideas on this point are contained in the following excerpts:

"Concrete geometry, in one form or another, may be a constant part of the work in arithmetic from the earliest years. The transition to strict demonstration should not be abrupt, but gradual. Quite a little of informal, but real, demonstration may be done in the last year or two before the secondary school, so that when the pupil nominally begins the study of demonstrative geometry in the secondary school he has already not only a large stock of geometric names and facts, but the spirit of demonstration well awakened" (p. 263).

"When should more formal geometry be begun? Whenever the pupil feels the need of it. Use intuition freely and prove what does not seem evident. The pupil may accept intuition as proof; that is not bad. Many great mathematicians have done so with happy results for the science. As long as he accepts intuitionally statements which are true, *go on*—things are progressing nicely. When his intuition fathers a falsity, *prove* to him that it is false (by concrete example or the like). This will arouse him to a sharp criticism of what constitutes a proof, *your* proof, and will do more to teach him the nature of a proof than dozens of routine demonstrations learned by rote, or even passively understood" (p. 264).

"The best course to be taken seems to be to base the work

* Playfair, 1813 (cf. Halsted's translation of the "Science Absolute of Space," by Bolyai, p. 67) and Thibaut, 1818 (cf. Killing, "Einführung in die Grundlagen der Geometrie," Vol. I, p. 7).

on *large* bodies of hypotheses, assuming many things which might be proved, but to make inferences from these hypotheses in strict accord with the laws of thought (logic). This does not mean that everything should be built up by strict deductive reasoning on the basis of the premises assumed, nor that the really deductive reasoning used should always be expressed in the most formalistic manner, but that when professing to be strict, the reasoning should really *be* so, and that there should always be a clear line drawn between what is proved and what is assumed as sufficiently plausible without proof, or accepted and used on the assertion of others (*e. g.*, teacher) that it can be proved. Absolute truth towards the pupil is the first essential.

"Let the idea of building up a system of propositions rigorously deduced from a set of irréducible assumptions (axioms) be abandoned, but let the distinction between what constitutes a *proof* in mathematics and what does not be all the more emphasized" (p. 123).

"The recent researches on axioms explode, once for all, all hope of teaching the child a logically perfect geometry in which all the results are deduced from a set of irreducible first principles, and *with it goes the only justification for adhering longer to the semblance of such a rigorous deductive system*" (reviewer's italics, p. 199).

Most mathematicians will be in hearty sympathy with nearly all of this, but the italicized sentence seems to the reviewer to be a case of *non sequitor*. Can we not apply, as it were, the "method of successive approximations"? Let the first approximation in the grammar schools have the object of familiarizing the pupil with "geometric names and facts" and "the spirit of demonstration" if possible. A second approximation with the same object may be desirable in the high school. The third approximation, which may legitimately, from the historical point of view, be called demonstrative geometry, might be built upon a large body of dependent axioms, consistent of course, and complete except for considerations which seem trivial to the pupil, *e. g.*, of three points on a line one is between the other two. The pupil may thus obtain the important notion, far too important to be sacrificed, of a "logically perfect geometry," even while realizing, perhaps not until late in the course, that his own study is incomplete in detail. If solid geometry is taken

in college there is opportunity for a fourth approximation, while the final approximation would be a graduate course for the specialist in the foundations of geometry.

The Playfair-Thibaut proof mentioned above might well be used in introductory work, but in demonstrative geometry, the introduction of a new axiom, either implicitly or explicitly, to prove but a single theorem which may be based upon an axiom frequently used, seems inadvisable; especially when, as in the case in hand, the new axiom is dependent upon the other in such a hidden fashion that specialists have failed to perceive the dependence.

Three short chapters on Preparation of teachers: mathematical clubs, The material equipment, and The curriculum in mathematics, separate the consideration of general pedagogical questions from the discussion of particular subjects. In the first one notes with some surprise the absence of projective geometry from the minimum list of subjects with which the teacher should be familiar. The last of the three emphasizes the desirability of teaching algebra and geometry simultaneously, and considers the co-ordination of mathematics with physics, a point that may easily be carried too far.

Chapter XI contains a discussion of definitions and axioms, showing clearly the distinction between different classes of mathematical terms and various points of view of the axioms. Not all will agree with the author's conclusion "to *use* the axiom when needed, perhaps without formulation, simply with a tacit or open 'of course'" (p. 200).

The following three chapters are devoted to the teaching of arithmetic, geometry, and algebra, respectively. The first of these, covering fifty-four pages, the longest chapter in the book, is the last to which many mathematicians would turn, and yet it is one of the most interesting. It contains many valuable suggestions which apply as well to other branches than arithmetic, and it is worth reading if only for the delightful story of the boy who obtained assistance from his parents. The teacher of arithmetic should read with especial care the pages on the algebraic and geometric sides of arithmetic. "It seems desirable that the study of geometric form be carried throughout the work in arithmetic with increasing thoroughness as the years pass, that the elements of literal arithmetic be introduced in such

measure as they can be utilized, the whole to be one coherent subject,—*arithmetic*" (p. 248).

(*To be continued.*)

## NOTES AND NEWS.

THOSE MEMBERS OF THE ASSOCIATION who have not paid their dues up to and including this year which ends November 30, 1909, need not expect to receive further copies of THE MATHEMATICS TEACHER. Let every member who has not already done so send their dues at once to the Treasurer, Miss Emma H. Carroll, Philadelphia High School for Girls, Philadelphia, Pa.

REPORT OF THE TWELFTH MEETING OF THE ASSOCIATION AT SYRACUSE UNIVERSIITY, APRIL 10, 1909.—*Committees.*—Reports of progress from the Committee on Algebra Syllabus and the Committee on Mathematics in Continuation Schools were read and approved and the committees were continued.

The Committee on Systematic Association Work was discharged.

*Amendments.*—Paragraph I. of Section II. was amended to read: "The Annual meeting shall be held at a time and place to be selected by the council."

It was announced that an amendment giving certain financial power to the editorial board of THE MATHEMATICS TEACHER, and providing for the term of office of the editors, would be voted on at the fall meeting.

*Next Meeting.*—The next meeting will be held at the College of the City of New York, at a date not yet determined.

The following new members were elected April 10, 1909:

S. BEAMAN, VERA H., A.B.; Gouverneur, N. Y.
P. BOWER, MARY ISABEL, Sc.B.; High School, Ridley Park, Pa.
S. BROWN, CLAUDE N., A.B.; Principal of High School, Pulaski, N. Y.
S. BROWN, JENNIE MAY, B.S.; High School, Bradford, Pa., 194 Jackson Ave.
S. BROWN, SUSIE D., M.Ph.; North Side High School, Syracuse, 1500 Park St.
S. COLEMAN, NELSON L., A.B.; High School, Binghamton, N. Y., 106 Prospect St.
S. ELDRED, ARVIE, A.B.; High School, Troy, N. Y.
S. GEORGIA, ANNA C., A.B.; Ellicottville High School, Sidney, N. Y.